A Simplification of a Step in Lin's Proof of the Structural Controllability Theorem

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Abstract

Structural controllability has been successfully applied to various actual physical systems, which attracts an increasing number of scholars to devoting to the relevant theoretical or practical research. A normal first step for the beginners in this area is to read Lin's seminal paper on structural controllability [1]. However, the equivalence proof of the second and third conditions of structural controllability theorem in Lin's paper is redundant, which hinders readers' understanding of Lin's work. In this tutorial, we provide an illustration of Lin's paper and simplify the proof mentioned above to help beginners understanding Lin's paper.

Index Terms

Structurally Controllability, Graph Theory

I. Introduction

Controllability is a fundamental concept in control theory, which reflects whether we can control a system as we desire. To verify the controllability of a linear time-invariant control system

$$\dot{x} = Ax + Bu, \tag{I.1}$$

where normally the entries in $A \in \mathcal{R}^{n\times n}$ represent the parameters in a dynamic system, such as the mass of a mass-spring-damper system or the interaction relationships between nodes and strengths of edges of a network, such as the connection in a social network and the entries in $B \in \mathcal{R}^{n\times m}$ represent the intensity and relationship on how the states (or the nodes) are directly affected by the inputs, it is required to know the exact value of every entry in the matrices A and B. However, in practical situations, the exact values of the entries A and B are often hard to obtain due to measurement errors or a lack of knowledge [1, 2].

In order to make actual physical systems applicable for controllability tests, Lin introduced the notion "Structural Controllability" in 1974 [1]. This notion focuses on a type of linear systems called "linear structured systems",

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which is defined in the sense that every entry in the linear structured system $(\hat{A}, \hat{B})$ is either a fixed zero or a free parameter including zero [10]. The systems with the identical locations of fixed zeros are called with the same structure. If a structured system is controllable after assigned values to the free parameters, then we call this structured system structurally controllable. As pointed out in [1, 2, 6], except some special values, a structurally controllable system is controllable for nearly all the values assigned to the structured system.

Since structural controllability analysis is a powerful tool to verify the controllability of actual physical systems, it has been applied to various fields, ranging from biological systems [2–4] to network security [5]. In turn, the wide application of structural controllability is also attracting increasing attention to the research on its theoretical development. To begin the research of structural controllability, whether for theoretical or practical research, a normal first step is to read Lin's paper [1]. Unfortunately, Lin adopted a redundant method to prove the equivalence of the second and third conditions of the structural controllability theorem (illustrated in the following section), which often impedes one's understanding of his paper. To help researchers better understanding Lin's paper, we wrote this paper and provided a more straightforward proof regarding the equivalence of the second and third conditions of Lin's theorem by employing a perfect matching approach.

A. *Related work*

Lin's work [1] only considered a control system with a single input, the result of which was extended to a multi-input case in [6, 7]. In [8], the author proposed a graph theoretic interpretation for the multi-input case structural controllability conditions. The structural controllability theorem of parameterized linear system was introduced in [11] and the corresponding graph theoretical conditions were given in [12]. In a parameterized linear system, some free parameters are specified to be identical. The research on structural controllability of parameterized linear systems allows one to add more structural information of a system into the structural controllability investigation.

Another issue on structural controllability is minimal inputs problem, where the core question is to find the minimum number of input nodes to make the system structurally controllable. In [2], the authors determined the minimum number of input nodes by applying a perfect matching method. In [9], the author solved a class of minimum input problem with a set of states which can not be affected by inputs.

There is also a related notion "strong structural controllability", where if assigned arbitrary nonzero values to the free parameters of a linear structured control system, the system remains controllable, then we call this structured system strongly structurally controllable [13]. It is obvious to see that a control system being controllable is a necessary condition for it to be strongly structurally controllable. The notion strong structural controllability is out of the scope of this paper. Interested readers may refer to [13–17].

II. Lin's Theorem

In [1], the author adopted a graph theoretical method to prove the structural controllability theorem, which is illustrated below [1, 2, 18, 19].

Given a matrix pair (A, b), where $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$, it can be represented by a directed graph $\mathcal{G}(A, b)$, where the node set is denoted by $\mathcal{N}$, and the edge set is denoted by $\mathcal{E}$. Every node corresponds to one column in

Fig. 1. A graph of a stem with four nodes.

the matrix $[A, b]$. In particular, the node corresponds to the column of b is called the "origin" node of the graph $\mathcal{G}$. Every free entry a_{ij} in the matrix A corresponds to one edge starting at node j and ending at node i. Similarly, a free entry b_i in the matrix b corresponds to one edge starting at node $n+1$ to node i.

We define a *walk* as a sequence $v_1, v_2, \cdots, v_t$ of nodes with directed edges $(v_1, v_2), (v_2, v_3), \cdots, (v_{t-1}, v_t)$. A *directed path* is defined as a walk if all of its nodes are distinct and a *directed cycle* is defined as a walk if all of its nodes are distinct and $v_1 = v_t$.

Next, we define several special graph components in the graph $\mathcal{G}(A, b)$.

Definition 1 (stem) : A stem is a directed path starting from the origin node. The corresponding matrices are

$$A = \begin{bmatrix} 0 & a_{12} & 0 & \cdots & 0 \\ 0 & 0 & a_{23} & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \cdots & a_{(n-1)n} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ a_{n(n+1)} \end{bmatrix}$$

and in Fig. 1, we show a four-node stem graph.

Definition 2 (bud). A bud is a directed cycle with an extra edge e, which ends, not beginning, in a node of the directed cycle. The extra edge is called the distinguished edge of the bud. The corresponding matrices are

$$A = \begin{bmatrix} 0 & a_{12} & 0 & \cdots & 0 \\ 0 & 0 & a_{23} & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \cdots & a_{(n-1)n} \\ a_{n1} & 0 & 0 & \cdots & 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ a_{n(n+1)} \end{bmatrix}$$

and in Fig. 2, we present a four-node bud graph.

Definition 3 (inaccessible). If there does not exist a directed path from the origin to one node, then we call the node "inaccessible".

Remark 1: The presence of "inaccessible" node means there exists a permutation matrix P, such that

$$P^{-1}AP = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}; P^{-1}b = \begin{bmatrix} 0 \\ b_2 \end{bmatrix},$$

which by simple calculation, one can find that the matrix pair

$$\left(\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} 0 \\ b_2 \end{bmatrix} \right)$$

is not controllable [2, 6].

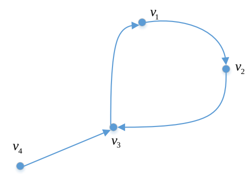

Fig. 2. A graph of a bud with four nodes.

Definition 4 (dilation). If for two node sets S and $T(S)$, $|S| > |T(S)|$, then we call there exists a dilation, where $|\cdot|$ denotes the cardinality of a set. Here, $S \subseteq \mathcal{N}_A$ and the neighbor node set $T(S)$ is defined as the node set of all the nodes v_j such that there exists an oriented edge node v_j to one node $v_i \in S$. Note that the origin node can only belong to a node set $T(S)$, not in a node set S.

In fact, a dilation is closely related to the notion "perfect matching" and the relationship will be illustrated later.

Remark 2: The presence of dilation means that the generic rank of matrix $[A, b]$ is less than n. The generic rank of a structure matrix M is defined to be the maximum rank that a matrix can reach as as a function of all the free parameters in M [2, 6].

Definition 5 (cactus). The graph $\mathcal{G}(A, b)$ is a cactus if and only if $\mathcal{G}(A, b) = S \cup B_1 \cup B2 \cup \cdots \cup B_t$, where S is a stem and B_i is a bud and for every $i = 1, 2, \cdots, t$, the extra edge e_i of B_i starts at $S \cup B_1 \cup B2 \cup \cdots \cup B_{t-1}$. Moreover, e_i is the only edge which belongs to both bud B_i and $S \cup B_1 \cup B2 \cup \cdots \cup B_{t-1}$. A graph of a cactus is shown in Fig. 3.

Now, we can present the structural controllability theorem in Lin's paper [1, 2].

Theorem 1:

1) A linear control system (A, b) is structurally controllable.
2) a) The directed graph $\mathcal{G}(A, b)$ contains no inaccessible nodes.
   b) The directed graph $\mathcal{G}(A, b)$ contains no dilation.
3) The directed graph $\mathcal{G}(A, b)$ is spanned by a cactus.

Remark 3: A directed graph $\mathcal{G}(\mathcal{N}_1, \mathcal{E}_1)$ spanned by a directed graph $\mathcal{G}(\mathcal{N}_2, \mathcal{E}_2)$ is defined as $\mathcal{N}_1 = \mathcal{N}_2$ and $\mathcal{E}_2 \subseteq \mathcal{E}_1$.

In Lin's paper, the equivalence proof of the second and third conditions is redundant and in the next section, we provide a perfect matching approach, which can simplify the proof.

III. Perfect Matching Approach

In this section, we first introduce the definition of bipartite graph and perfect matching. Then we present the simplified proof.

A. Bipartite Graph and Perfect Matching

For a directed graph, a matching is an edge subset M if no two edges in M share a common starting node or a common ending node. A node is called matched if it is an ending node of an edge in the matching. Otherwise, it

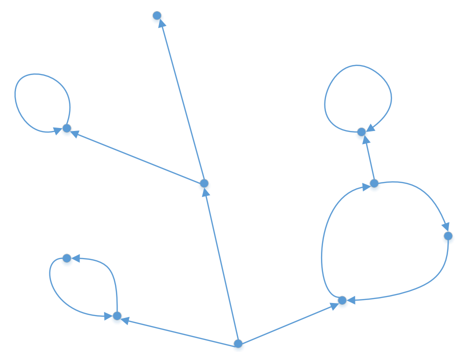

Fig. 3. A graph of a cactus.

is unmatched [2].

A matching with maximum cardinality is called a maximum matching. A maximum matching is perfect if all nodes are matched.

A convenient way to find the maximum matching in a digraph is to represent the digraph with a bipartite graph. Below we first state the definition of bipartite graph and then we present the Hall's theorem which is a method to find if there is a perfect matching in the bipartite graph.

Definition 6 (bipartite graph). A graph $\mathcal{G}(\mathcal{N},\mathcal{E})$ is called bipartite if there is a partition of $\mathcal{N}$ into the disjoint subsets $\mathcal{N}=\mathcal{N}^+\cup\mathcal{N}^-$, such that every edge $e\in\mathcal{E}$ connects some vertex in $\mathcal{N}^+$ to some vertex in $\mathcal{N}^-$.

Similar to the definition in a directed graph, a bipartite graph has a perfect matching if all the nodes in $\mathcal{N}^-$ are matched. In fig. 4, we show a matching in a bipartite graph.

Next, we present Hall's Theorem which is a method to test whether there exists a perfect matching in a bipartite graph.

Theorem 2: (Hall's Theorem [20]) A bipartite graph $\mathcal{G}(\mathcal{N}^+,\mathcal{N}^-,\mathcal{E})$ has a perfect matching if and only if for every subset $S\subseteq\mathcal{N}^-, |T(S)|\geqslant|S|$, where $T(S)\subseteq\mathcal{N}^+$ is the neighbor node set of S.

By employing Hall's Theorem, we can rewrite the condition 2) b) of theorem 1 as:

The bipartite graph of the directed graph $\mathcal{G}(A,b)$ has a perfect matching.

B. *Proof Reformulation*

In this section, we will provide a simplified proof demonstrating the equivalence of the second and third conditions of Lin's structural controllability theorem.

Proof:

First, we prove $3\rightarrow 2$.

From the definition of a cactus graph, it is obvious that every node in the directed graph $\mathcal{G}(\mathcal{N},\mathcal{E})$, which is spanned by a cactus graph $\mathcal{G}(\mathcal{N},\mathcal{E}_1)$, is accessible. As a result, what we need to do now is to prove the directed graph $\mathcal{G}(\mathcal{N},\mathcal{E})$ has a perfect matching.

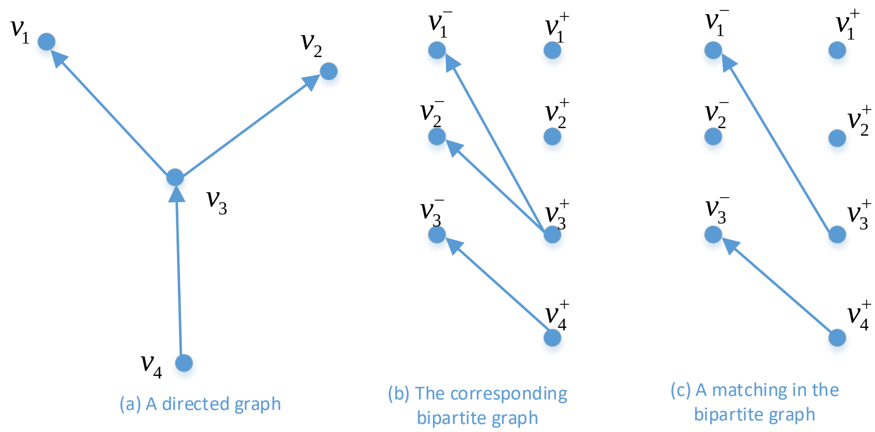

Fig. 4. A matching in a bipartite graph.

Delete the edges of the $\mathcal{G}(\mathcal{N},\mathcal{E})$ so that the remaining graph is the the cactus graph $\mathcal{G}(\mathcal{N},\mathcal{E}_1)$. Then we delete the distinguished edges of the graph $\mathcal{G}(\mathcal{N},\mathcal{E}_1)$ and denote the remaining graph by $\mathcal{G}(\mathcal{N},\mathcal{E}_2)$, which is composed only by one or zero stem and several directed cycle graphs. One can easily verify that $\mathcal{G}(\mathcal{N},\mathcal{E}_2)$ has a perfect matching. Thus, the bipartite graph of the directed graph $\mathcal{G}(\mathcal{N},\mathcal{E})$ also has a perfect matching.

Next, we show $2 \to 3$.

Because the bipartite graph of the directed graph $\mathcal{G}(\mathcal{N},\mathcal{E})$ has a perfect matching, which is denoted by $\hat{\mathcal{E}}$, we can construct a graph $\mathcal{G}(\mathcal{N}_1,\hat{\mathcal{E}}_1)$ by the following steps. First add the origin node $v_{n+1}^+ \in \mathcal{N}^+$, an edge $e_1 \in \hat{\mathcal{E}}$, which is incident with the node v_{n+1}^+, and the node $v_1^- \in \mathcal{N}^-$, which is the other node that the edge e_1 connects, to the sets $\mathcal{N}_1$ and $\hat{\mathcal{E}}_1$, respectively. Then add the node $v_1^+ \in \mathcal{N}^+$, an edge $e_2 \in \hat{\mathcal{E}}$, which is incident with the node v_1^+, and the node $v_2^- \in \mathcal{N}^-$, which is the other node that the edge e_2 connects, to the sets $\mathcal{N}_1$ and $\hat{\mathcal{E}}_1$, respectively. Next repeat the above step until there is no matching node in $\mathcal{N}^-$ for the node $v_i^+ \in \mathcal{N}^+$. Thus, $\mathcal{N}_1 = \{v_1,\cdots,v_i,v_{n+1}\}$ and $\hat{\mathcal{E}}_1 = \{e_1,\cdots,e_i\}$. By definition, the graph $\mathcal{G}(\mathcal{N}_1,\hat{\mathcal{E}}_1)$ is a stem.

Then we demonstrate the graph $\mathcal{G}(\mathcal{N}-\mathcal{N}_1,\hat{\mathcal{E}}-\hat{\mathcal{E}}_1)$ is composed of several directed cycles. Here we only show why one directed cycle in the graph $\mathcal{G}(\mathcal{N}-\mathcal{N}_1,\hat{\mathcal{E}}-\hat{\mathcal{E}}_1)$ exists. The reason why other directed cycles in the graph $\mathcal{G}(\mathcal{N}-\mathcal{N}_1,\hat{\mathcal{E}}-\hat{\mathcal{E}}_1)$ exist can be proved similarly.

Let $\mathcal{N}_2$ and $\mathcal{E}_2$ denote a node set and an edge set, respectively. Add a node $v_{i+1}^+ \in \mathcal{N}^+ - \mathcal{N}_1^+$, an edge $e_{i+1} \in \hat{\mathcal{E}} - \hat{\mathcal{E}}_1$, which is incident with the node v_{i+1}^+, and the node $v_{i+2}^- \in \mathcal{N}^+ - \mathcal{N}_1^+$, which is the other node that the edge e_{i+1} connects, to $\mathcal{N}_2$ and $\hat{\mathcal{E}}_2$, respectively. Then add a node $v_{i+2}^+ \in \mathcal{N}^+ - \mathcal{N}_1^+$, an edge $e_{i+2} \in \hat{\mathcal{E}} - \hat{\mathcal{E}}_1$, which is incident with the node v_{i+2}^+, and the node $v_{i+3}^- \in \mathcal{N}^+ - \mathcal{N}_1^+$, which is the other node that the edge e_{i+2} connects, to $\mathcal{N}_2$ and $\hat{\mathcal{E}}_2$, respectively. Repeat the above step until $v_{i+j}^+ = v_{i+1}^-$. By definition, the graph $\mathcal{G}(\mathcal{N}_2,\hat{\mathcal{E}}_2)$ is a directed cycle.

Since every node is accessible, there exist edges $e \in \mathcal{E} - \hat{\mathcal{E}}$ so that there is a directed path from the origin to every node and we thus obtain a cactus graph, which means $\mathcal{G}(\mathcal{N},\mathcal{E})$ is spanned by a cactus graph.

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IV. Conclusion

In this paper, we reviewed the notion structural controllability and Lin's structural controllability theorem. We also provided a simplified proof of the equivalence of the second and third conditions of Lin's structural controllability theorem.

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